

**MECÂNICA E MODELAÇÃO COMPUTACIONAL**  
Mestrado Integrado em Engenharia Biomédica

1ª Época

Ano Lectivo de 2010/2011

15/Janeiro/2011

- O Exame é sem consulta. O formulário está anexo a este exame.
- Não são permitidos computadores pessoais ou telemóveis.
- Todas as folhas do exame deverão ser identificadas.
- A duração do exame é de 2 horas e 30 minutos.

**Problema I (6 val.)**

Considere o esquema da figura 1a, o qual representa um fémur onde foi colocada uma haste femoral em titânio. A haste está completamente solidária com o osso. Admita que se quer estudar o comportamento à torção da região abaixo da secção E-E. Para tal considerou-se o modelo simplificado apresentado na figura 1b e admitiu-se que o momento torsor aplicado é de  $T=1000 \text{ Nm}$ . Considere o módulo de rigidez à torção do osso compacto e do titânio são  $G_{osso}=10 \text{ GPa}$  e  $G_{haste}=55 \text{ GPa}$ , respectivamente. Considere ainda que o diâmetro exterior do osso compacto é  $30 \text{ mm}$  e o diâmetro da haste é  $18 \text{ mm}$ , e o comprimento  $L=100 \text{ mm}$

Calcule:

- a) A tensão de corte máxima no osso compacto na secção A-A e na secção B-B. (2,5 val.)
- b) A tensão de corte máxima no titânio. (1,0 val.)
- c) O ângulo de torção na extremidade C. (1,0 val.)
- d) A tensão normal máxima na secção B-B e indique o plano em que ocorre. (1,5 val.)

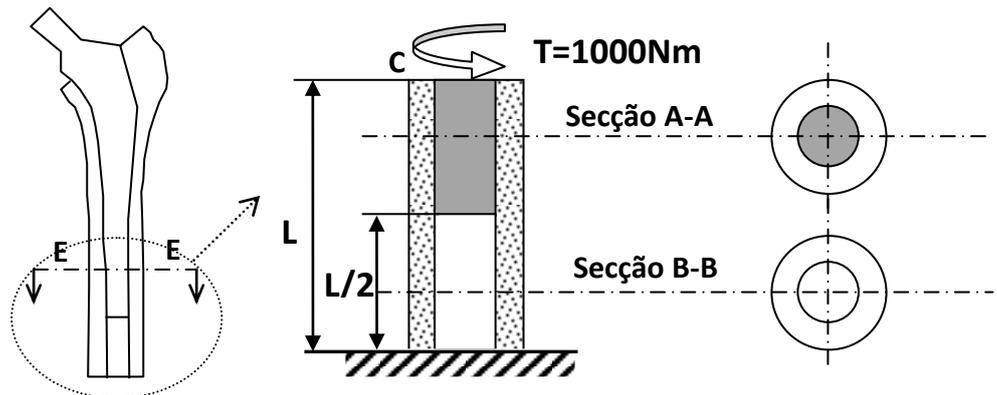


Figura 1-a

Figura 1-b

**Problema II (3 val.)**

De forma a permitir o acesso ao interior de um tubo quadrado com parede de espessura  $6 \text{ mm}$ , a porção BD de um lado do tubo foi removida. Sabendo que a carga no tubo é equivalente a duas forças iguais e opostas de  $60 \text{ kN}$  que actuam nos centros geométricos A e E das extremidades do tubo, determine:

- a) A tensão máxima na secção a-a. (2,0 val.)
  - b) A tensão no ponto F. (1,0 val.)
- Considere o centróide da secção em C e  $I_z=2 \times 10^6 \text{ mm}^4$ .

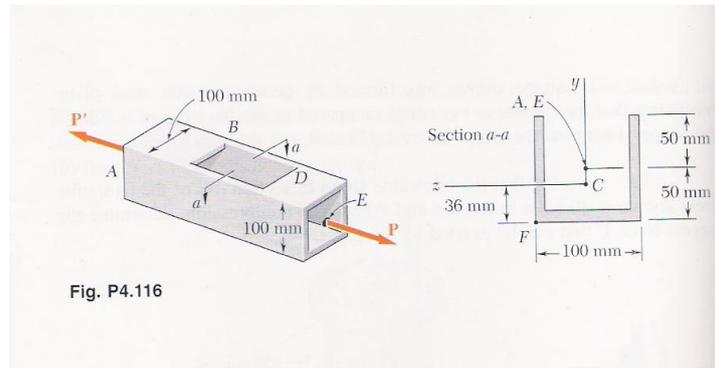
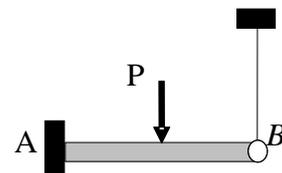


Fig. P4.116

### Problema III (6 val.)

Considere a estrutura da figura formada por uma **viga** horizontal de comprimento  $L=2m$ , módulo de Young  $E=70GPa$ , momento de inércia  $I=5 \times 10^{-6} m^4$ , encastradas na extremidade esquerda, ligada na extremidade direita a um **barra** vertical feito do mesmo material e comprimento e secção  $A=1 \times 10^{-3} m^2$ , A viga está sujeita a uma carga  $P=2kN$  a meio vão.

- Determine a força de tracção na barra. (2,0 val.)
- Determine o deslocamento vertical do ponto B. (1,0 val.)
- Determine as reacções em A. (1,0 val.)
- Trace os diagramas de esforço transversal e momento flector para a viga (2,0 val.)

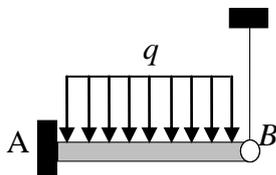


### Problema IV (5 val.)

Considere a estrutura do problema III, mas neste caso com a viga está sujeita a uma carga distribuída constante  $q = 1kN/m$ .

Considerando que para resolver o problema pelo Método dos elementos Finitos se utiliza apenas um elemento finito de viga e um elemento de barra:

- Estabeleça o modelo de elementos finitos e respectiva matriz de conectividade. (0,5 val)
- Determine a matriz de rigidez global da estrutura. (1,0 val)
- Determine a vector de força global da estrutura. (1,0 val)
- Identifique as condições de fronteira essenciais do problema no modelo e obtenha a solução do problema (os graus de liberdade não nulos). (1,5 val)
- Admitindo que os graus de liberdade para o elemento da viga são  $\left\{ 0, 0, \frac{qL^4}{8EI}, -\frac{qL^3}{6EI} \right\}$ , determine a distribuição do momento flector . Comente a precisão da solução obtida. (1,0 val)



# COMPUTATIONAL MODELLING IN MECHANICS

Integrated Master (MSc) in Biomedical Engineering

1<sup>st</sup> Exam

2010/2011

15/January/2011

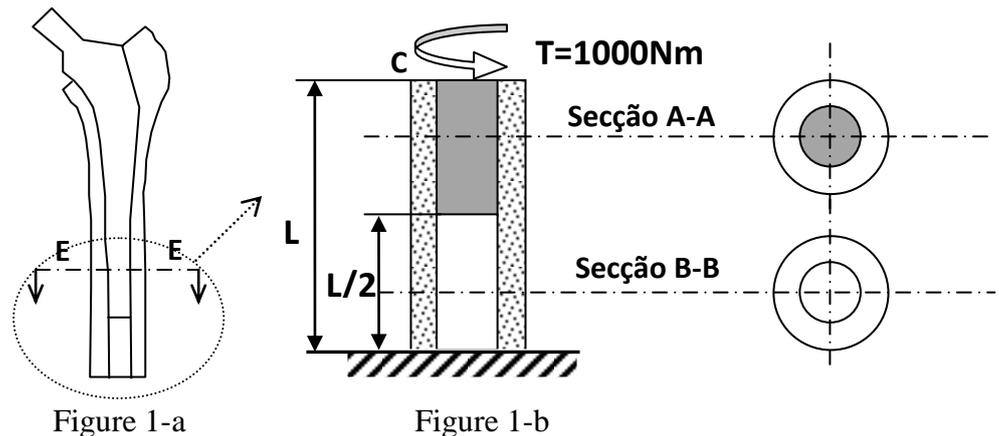
- It is a close book exam. A list of formulas is attached to this exam.
- Cellular phones and personal computers are not allowed.
- Please indentify all paper sheets.
- The exam duration is 2 hours and 30 minutes.

## Problem I (6.0)

Figure 1a shows an implanted femur with a titanium femoral stem. Stem and bone are fully bonded. The goal is to study the behavior of the region below the section E-E, when the femur is subject to torsion. To perform the analysis a simplified model is used as shown in figure 1b. The magnitude of the torque is  $T=1000\text{ Nm}$ , the shear modulus is  $G_{bone}=10\text{ GPa}$  for bone and  $G_{stem}=55\text{ GPa}$  for the stem. The external diameter of compact bone is  $30\text{ mm}$ , the diameter of the stem is  $18\text{ mm}$ , and the length of the portion below the section E-E is  $L=100\text{ mm}$ .

Determine:

- The maximum shear stress in compact bone at section A-A and at section B-B. (2.5)
- The maximum shear stress for the titanium stem. (1.0)
- The angle of twist at C. (1.0)
- The maximum normal stress at section B-B. Identify the plane where it occurs. (1.0)



## Problem II (3.0)

In order to provide access to the interior of a hollow square tube of 6-mm wall thickness, the portion CD of one side of the tube has been removed (see figure 2). Knowing that the loading of the tube is equivalent to two equal and opposite 60 kN forces acting at the geometric centers A and E of the ends of the tube, determine:

- the maximum stress in section a-a. (2.0)
- the stress at point F. (1.0)

Given: the centroid of the cross section is at C and  $I_z=2\times 10^6\text{ mm}^4$ .

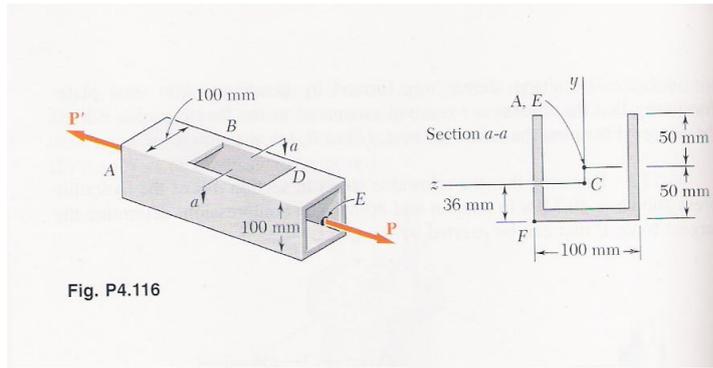
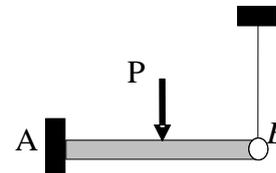


Figure 2

**Problema III (6 val.)**

A horizontal **beam** with length  $L=2m$ , Young modulus  $E=70GPa$ , and moment of inertia  $I=5 \times 10^{-6} m^4$ , is clamped on the left end and attached to a vertical **bar** at the right end. The bar is made of the same material as the beam, with equal length and cross sectional area  $A=1 \times 10^{-3} m^2$ . The beam is subject to an applied load  $P=2kN$  at midspan.

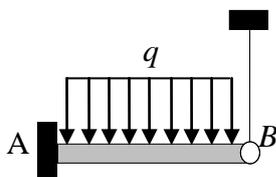
- e) Determine the tension in the bar. (2.0)
- f) Determine the vertical displacement at B. (1.0)
- g) Determine the reactions at A. (1.0)
- h) Draw the shear and bending-moment diagrams for the beam (2.0).



**Problema IV (5 val.)**

Consider the structure of problem III, with a distributed load of  $q = 1kN/m$ . To obtain the solution by the Finite Element Method, a beam and a bar element are used.

- a) Define the finite element model and the respective matrix of connectivity. (0.5)
- b) Determine the global stiffness matrix (1.0)
- c) Determine a global load vector. (1.0)
- d) Indicate the essential boundary condition for this structure and determine the solution for the problem (the non-zero degree of freedom). (1.5)
- e) Assuming that the degrees of freedom for the beam element are  $\left\{ 0, 0, \frac{qL^4}{8EI}, -\frac{qL^3}{6EI} \right\}$ , determine the distribution of bending-moment. Comment the accuracy of the obtained solution. (1.0)



## FORMULÁRIO:

$$\sigma = \frac{P}{A} \quad \sigma = \lim_{\Delta A \rightarrow 0} \frac{\Delta F}{\Delta A} \quad \tau = \lim_{\Delta A \rightarrow 0} \frac{\Delta V}{\Delta A} \quad \tau_{xy} = \tau_{yx} \quad \sigma = \frac{P}{A_0} \cos^2 \theta$$

$$\tau = \frac{P}{A_0} \cos \theta \sin \theta \quad \varepsilon = \frac{d\delta}{dx} \quad \sigma = E\varepsilon \quad \varepsilon_T = \alpha \Delta T \quad \delta = \frac{PL}{AE}$$

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} \quad \delta = \int_0^L \frac{P}{AE} dx$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} \quad \varepsilon_y = -\nu \frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} \quad \varepsilon_z = -\nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$\tau_{xy} = G\gamma_{xy} \quad \tau_{yz} = G\gamma_{yz} \quad \tau_{xz} = G\gamma_{xz} \quad G = \frac{E}{2(1+\nu)}$$

$$\gamma = \frac{\rho\phi}{L} \quad \tau = \frac{T\rho}{J} \quad \phi = \frac{TL}{JG} \quad \phi = \sum_i \frac{T_i L_i}{J_i G_i} \quad \phi = \int_0^L \frac{T}{JG} dx$$

$$\frac{1}{\rho} = \frac{M}{EI} \quad \frac{1}{\rho'} = \frac{\nu}{\rho} \quad \sigma_x = -\frac{My}{I} \quad \sigma_x = \frac{P}{A} - \frac{My}{I}$$

$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y} \quad \tan \phi = \frac{I_z}{I_y} \tan \theta \quad \tau = \frac{VQ}{It} \quad \frac{dV}{dx} = -w$$

$$V_D - V_c = - \int_{x_c}^{x_D} w dx \quad \frac{dM}{dx} = V$$

$$M_D - M_c = \int_{x_c}^{x_D} V dx \quad \frac{1}{\rho} = \frac{M x}{EI} \quad \frac{d^2 y}{dx^2} = \frac{M x}{EI} \quad F = -kx$$

$$K_{ij}^e = \int_0^{h^e} EA \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx \quad F_i^e = \int_0^{h^e} f \phi_i dx \quad [K_{ij}^e] = \frac{E^e A^e}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad F_i^e = \frac{f^e L^e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\phi_1 = 1 - \frac{x}{h^e} \quad \phi_2 = \frac{x}{h^e} \quad K_{ij}^e = \int_0^{h^e} EI \frac{d^2 \phi_i}{dx^2} \frac{d^2 \phi_j}{dx^2} dx \quad F_i^e = \int_0^{h^e} q \phi_i dx$$

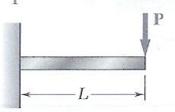
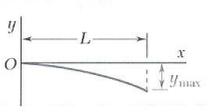
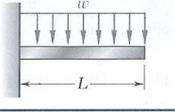
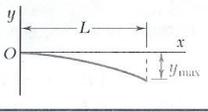
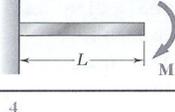
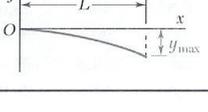
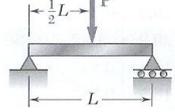
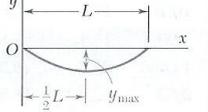
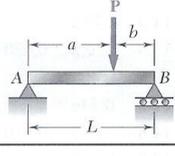
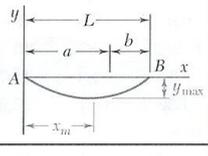
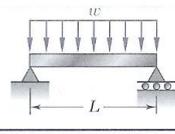
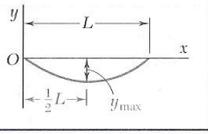
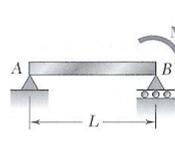
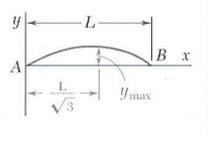
$$\phi_1 = 1 - 3\left(\frac{\bar{x}}{h^e}\right)^2 + 2\left(\frac{\bar{x}}{h^e}\right)^3 \quad \phi_2 = -\bar{x} \left(1 - \frac{\bar{x}}{h^e}\right)^2$$

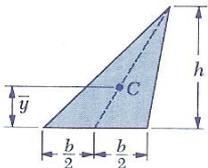
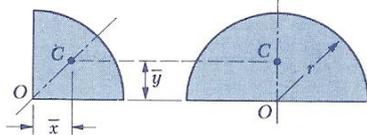
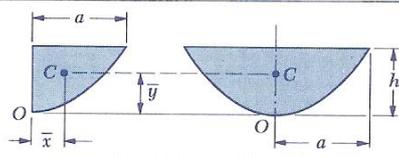
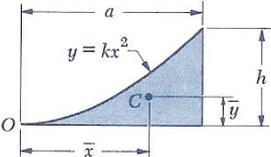
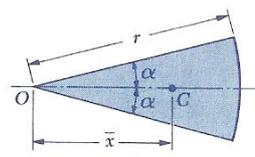
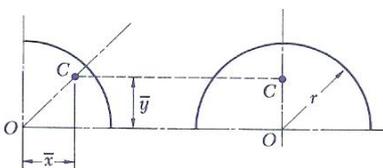
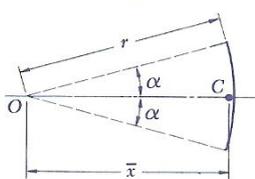
$$\phi_3 = 3\left(\frac{\bar{x}}{h^e}\right)^2 - 2\left(\frac{\bar{x}}{h^e}\right)^3 \quad \phi_4 = -\bar{x} \left[ \left(\frac{\bar{x}}{h^e}\right)^2 - \frac{\bar{x}}{h^e} \right]$$

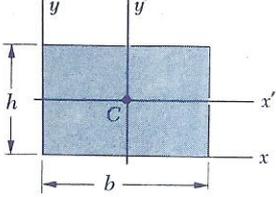
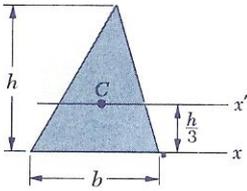
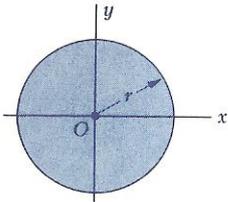
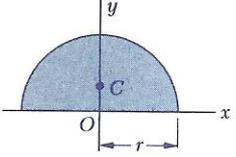
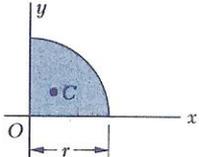
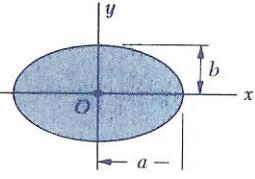
$$[K_{ij}^e] = \frac{2EI}{h_e^3} \begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix} \quad F_i^e = \frac{qh_e}{12} \begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix}$$

$$K_{\alpha\beta}^e = \int_{\Omega^e} k_{ij} \frac{\partial \phi_\alpha}{\partial x_i} \frac{\partial \phi_\beta}{\partial x_j} d\Omega \quad F_\alpha^e = \int_{\Gamma_e} q \phi_\alpha d\Gamma$$

$$\phi_1 = \frac{1}{4} (1 - \xi) (1 - \eta) \quad \phi_2 = \frac{1}{4} (1 + \xi) (1 - \eta) \quad \phi_3 = \frac{1}{4} (1 + \xi) (1 + \eta) \quad \phi_4 = \frac{1}{4} (1 - \xi) (1 + \eta)$$

Viga e carregamento	Linha elástica	Flèche máxima	Rotação na extremidade	Equação da linha elástica
1 		$\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y = \frac{P}{6EI}(x^3 - 3Lx^2)$
2 		$\frac{pL^4}{8EI}$	$-\frac{pL^3}{6EI}$	$y = -\frac{p}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$
3 		$\frac{ML^2}{2EI}$	$-\frac{ML}{EI}$	$y = -\frac{M}{2EI}x^2$
4 		$\frac{PL^3}{48EI}$	$\pm \frac{PL^2}{16EI}$	Para $x \leq \frac{1}{2}L$ : $y = \frac{P}{48EI}(4x^3 - 3L^2x)$
5 		Para $a > b$ : $\frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}EIL}$ em $x_m = \sqrt{\frac{L^2 - b^2}{3}}$	$\theta_A = -\frac{Pb(L^2 - b^2)}{6EIL}$ $\theta_B = +\frac{Pa(L^2 - a^2)}{6EIL}$	Para $x < a$ : $y = \frac{Pb}{6EIL}[x^3 - (L^2 - b^2)x]$ Para $x = a$ : $y = -\frac{Pa^2b^2}{3EIL}$
6 		$\frac{5pL^4}{384EI}$	$\pm \frac{pL^3}{24EI}$	$y = -\frac{p}{24EI}(x^4 - 2Lx^3 + L^3x)$
7 		$\frac{ML^2}{9\sqrt{3}EI}$	$\theta_A = +\frac{ML}{6EI}$ $\theta_B = -\frac{ML}{3EI}$	$y = -\frac{M}{6EIL}(x^3 - L^2x)$

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	$\alpha r^2$
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	$\pi r$
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$