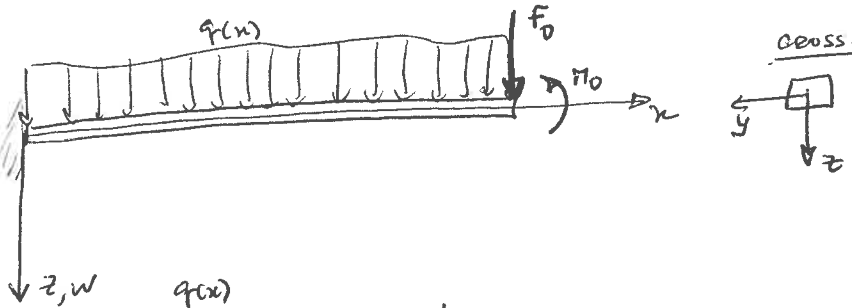
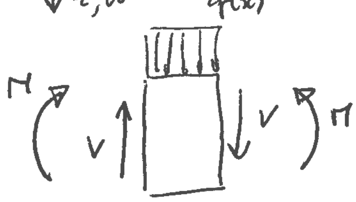
FINITE ELEMENT METHOD FOR BEAMS





$$\frac{dV}{dx} = -q(x)$$

$$\frac{dM}{dx} = V(x)$$

$$V - (V + \Delta V) - q(x)\Delta x = 0$$

$$-\Delta V = q(x)\Delta x$$

$$\frac{\Delta V}{\Delta x} = -q(x)$$

$$\Delta x \to 0 \Rightarrow \frac{dV}{dx} = -q(x)$$

$$M - (M + \Delta M) + V\Delta x + \underbrace{q\Delta x \times \frac{\Delta x}{2}}_{\to 0} = 0$$

$$-\frac{\Delta M}{\Delta x} = -V$$

$$\Delta x \to 0 \Rightarrow \boxed{\frac{dM}{dx} = V}$$

com este referencial with this reference system $-\frac{1}{\rho} = \frac{M}{EI}$ então thus,

$$M = -EI\frac{d^2w}{dx^2}$$ $\boxed{w \to y(x)}$ → in the M of Mat. book

So,

$$\frac{dM}{dx} = -EI\frac{d^3w}{dx^3} \quad \text{or} \quad \frac{dM}{dx} = \frac{d}{dx}\left(-EI\frac{d^2w}{dx^2}\right)$$

$$V = -EI\frac{d^3w}{dx^2} \qquad V = \frac{d}{dx}\left(-EI\frac{d^2w}{dx^2}\right)$$

$$\frac{dV}{dx} = \frac{d^2}{dx^2}\left(-EI\frac{d^2w}{dx^2}\right) \Rightarrow \boxed{\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q(x)}$$

$$\frac{dV}{dx} = -EI\frac{d^4w}{dx^4}$$

$$-q(x) = -EI\frac{d^4w}{dx^4} \Rightarrow \boxed{EI\frac{d^4w}{dx^4} = q(x)} \to \text{if } EI = \text{const.}$$

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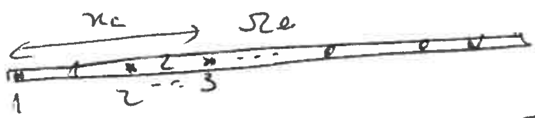
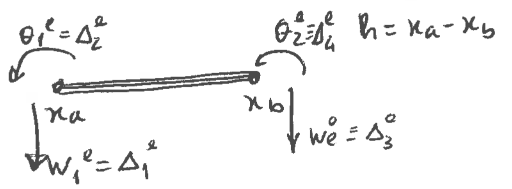
0. TO SOLVE BY THE F.E.M. WE HAVE TO DEFINE THE ELEMENT TO USE.

1. DISCRETIZATION OF THE DOMAIN



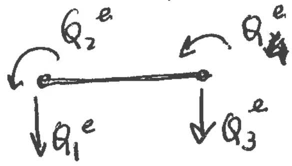
THE BEAM ELEMENT IS AN ELEMENT WITH TWO NODES:



BUT WITH TWO DEGREES OF FREEDOM PER NODE, THE VERTICAL DISPLACEMENT w AND THE ROTATION θ.

SO, THE VECTOR OF DEGREE OF FREEDOM IS $\Delta_i = \begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix}$ displac. rotation.

AND THE SECONDARY VARIABLES ARE:



$\begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$ forces / moments.

2 forces Q_1 and Q_3
2 moments Q_2 and Q_4

~~NOTES~~
Remark: The system for an element will be $[K^e]\{\Delta^e\} = \{f^e\} + \{Q^e\}$

2- WEAK FORM ~~AND~~

THE WEAK FORM FOR THE 4th ORDER EQUATION IS (see pag. 67 of the book)

$$\int_{x_e}^{x_{e+1}} \left[\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - q\right] v(x)\, dx = 0 \quad \text{for adm.}$$

$v(x)$ → weight function.

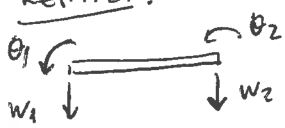
$$\int_{x_e}^{x_{e+1}} \left(EI \frac{d^2v}{dx^2}\frac{d^2w}{dx^2} - vq\right)dx + \underbrace{\left[v\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) - \frac{dv}{dx}EI\frac{d^2w}{dx^2}\right]_{x_e}^{x_{e+1}}}_{\text{terms related ~~to~~ the boundary condition.}} = 0$$

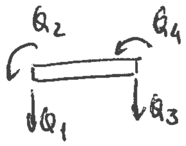
$$= \int_{x_e}^{x_{e+1}} \left(EI \frac{d^2v}{dx^2}\frac{d^2w}{dx^2} - vq\right)dx + v\underbrace{\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)}_{Q_3^e}\Bigg|_{x_{e+1}} - \frac{dv}{dx}\underbrace{EI\frac{d^2w}{dx^2}}_{Q_4}\Bigg|_{x_{e+1}} - v\underbrace{\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)}_{Q_1}\Bigg| + \frac{dv}{dx}\underbrace{EI\frac{d^2w}{dx^2}}_{Q_2}\Bigg|_{x_e}$$

ESSENTIAL BOUNDARY CONDITIONS → w AND $\frac{dw}{dx}$

NATURAL BOUNDARY CONDITIONS → $EI\frac{d^2w}{dx^2}$ → MOMENT
$\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)$ → TRANSVERSE FORCE

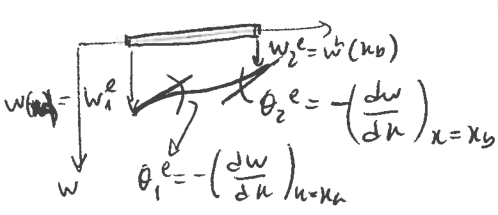
REMARK:





$M = -EI\frac{d^2w}{dx^2}$

$V = \frac{dM}{dx}$



$\theta = -\frac{dw}{dx}$

$$Q_1^e = \left[\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]\Bigg|_{x_e} = -V(x_e)$$

$$Q_2^e = \left(EI\frac{d^2w}{dx^2}\right)\Bigg|_{x_e} = -M(x_e)$$

$$Q_3^e = -\left[\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]\Bigg|_{x_{e+1}} = V(x_{e+1})$$

$$Q_4^e = -\left(EI\frac{d^2w}{dx^2}\right)\Bigg|_{x_{e+1}} = M(x_{e+1})$$

FINALLY THE WEAK FORM IS:

$$\int_{x_e}^{x_{e+1}}\left(EI\frac{d^2v}{dx^2}\frac{d^2w}{dx^2}-vq\right)dx-v(x_e)Q_1^e-\overbrace{\left(-\frac{dv}{dx}\right)\Big|_{x_e}}^{\theta(x_e)}Q_2^e-v(x_{e+1})Q_3^e$$

$$-\underbrace{\left(-\frac{dv}{dx}\right)\Big|_{x_{e+1}}}_{\theta(x_{e+1})}Q_4^e=0$$

3- INTERPOLATION FUNCTIONS (APROXIMATION FUNCTIONS)

- IN THIS CASE THE WEAK FORM REQUIRES FUNCTION WITH SECOND DERIVATIVE.
- THUS, THE APROXIMATION SHOULD BE TWICE DIFFERENTIABLE AND ~~IT~~ IT HAS TO SATISFY ~~THE~~ THE INTERPOLATION CONDITIONS:

$$W_h^e(x_e)=W_1^e\ ;\quad W_h^e(x_{e+1})=W_2^e\ ;\quad \theta_h^e(x_e)=\theta_1^e\quad \theta_h^e(x_{e+1})=\theta_2^e$$

THESE CONDITIONS IMPLY TO SATISFY ALSO THE CONTINUITY CONDITIONS.

SO, WE HAVE 4 CONDITIONS ~~FOR~~ A 2 NODE ELEMENT, AND THUS WE HAVE TO CHOSE A FOUR-PARAMETER POLYNOMIAL.

$$W(x)\simeq W_h^e(x)=C_1^e+C_2^e x+C_3^e x^2+C_4^e x^3$$ (the second derivative exists!)

→ it is a cubic polynomial.

NOW, WE HAVE TO DETERMINE THE PARAMETERS C_i^e, ~~F~~FUNCTION OF THE GENERALIZED DISPLACEMENTS Δ_i^e

$$\Delta_1^e\equiv W_h^e(x_e)\ ;\quad \Delta_2^e\equiv-\frac{dW_h^e}{dx}\Big|_{x=x_e}\ ;\quad \Delta_3^e\equiv W_h^e(x_{e+1})\ ;\quad \Delta_4^e\equiv-\frac{dW_h^e}{dx}\Big|_{x=x_{e+1}}$$

THUS:

$$\Delta_1^e=W_h^e(x_e)=C_1^e+C_2^e x_e+C_3^e x_e^2+C_4^e x_e^3$$

$$\Delta_2^e=-\frac{dW_h^e}{dx}\Big|_{x=x_e}=-C_2^e-2C_3^e x_e-3C_4^e x_e$$

$$\Delta_3^e=W_h^e(x_{e+1})=C_1^e+C_2^e x_{e+1}+C_3^e x_{e+1}^2+C_4^e x_{e+1}^3$$

$$\Delta_4^e=-\frac{dW_h^e}{dx}\Big|_{x=x_{e+1}}=-C_2^e-2C_3^e x_{e+1}-3C_4^e x_{e+1}$$

$$\begin{Bmatrix}\Delta_1^e\\ \Delta_2^e\\ \Delta_3^e\\ \Delta_4^e\end{Bmatrix}=\begin{bmatrix}1 & x_e & x_e^2 & x_e^3\\ 0 & -1 & -2x_e & -3x_e^2\\ 1 & x_{e+1} & x_{e+1}^2 & x_{e+1}^3\\ 0 & -1 & 2x_{e+1} & -3x_{e+1}^2\end{bmatrix}\begin{Bmatrix}C_1^e\\ C_2^e\\ C_3^e\\ C_4^e\end{Bmatrix}$$

SOLVING THIS SYSTEM WE OBTAIN:

$$W_h^e(x) = \sum_{j=1}^{4} \Delta_j^e \phi_j^e = \Delta_1^e \phi_1^e + \Delta_2^e \phi_2^e + \Delta_3^e \phi_3^e + \Delta_4^e \phi_4^e$$

$$\phi_1^e = 1 - 3\left(\frac{x - x_e}{h_e}\right)^2 + 2\left(\frac{x - x_e}{h_e}\right)^3$$

$$\phi_2^e = -(x - x_e)\left(1 - \frac{x - x_e}{h_e}\right)^2$$

$$\phi_3^e = 3\left(\frac{x - x_e}{h_e}\right)^2 - 2\left(\frac{x - x_e}{h_e}\right)^3$$

$$\phi_4^e = -(x - x_e)\left[\left(\frac{x - x_e}{h_e}\right)^2 - \frac{x - x_e}{h_e}\right]$$

$\rightarrow$ HERMITE CUBIC INTERPOLATION FUNCTIONS (CUBIC SPLINE)

(SHOW THE FUNCTION)

IN THE ELEMENT COORDINATE SYSTEM $\bar{x} = x - x_e$

$$\phi_1^e = 1 - 3\left(\frac{\bar{x}}{h_e}\right)^2 + 2\left(\frac{\bar{x}}{h_e}\right)^3$$

$$\phi_2^e = -\bar{x}\left(1 - \frac{\bar{x}}{h_e}\right)^2$$

$$\phi_3^e = 3\left(\frac{\bar{x}}{h_e}\right)^2 - 2\left(\frac{\bar{x}}{h_e}\right)^3$$

$$\phi_4^e = -\bar{x}\left[\left(\frac{\bar{x}}{h_e}\right)^2 - \frac{\bar{x}}{h_e}\right]$$

CHAPTER 5
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THESE FUNCTIONS SATISFY THE FOLLOWING INTERPOLATIONS PROPERTIES:

$$\phi_1^e(x_e) = 1 \qquad \phi_i^e(x_e) = 0 \quad (i \neq 1)$$

$$\phi_3^e(x_{e+1}) = 1 \qquad \phi_i^e(x_{e+1}) = 0 \quad (i \neq 3)$$

$$\left(-\frac{d\phi_2^e}{dx}\right)\bigg|_{x_e} = 1 \; ; \; \left(-\frac{d\phi_i^e}{dx}\right)\bigg|_{x_e} = 0 \quad (i \neq 2)$$

$$\left(-\frac{d\phi_4^e}{dx}\right)\bigg|_{x_{e+1}} = 1 \; ; \; \left(-\frac{d\phi_i^e}{dx}\right)\bigg|_{x_{e+1}} = 0 \quad (i \neq 4$$

(non dire) ??
ma anche quando
mostrare calcolo

4- FINITE ELEMENT MODEL

SUBSTITUTING THE APROXIMATION $w_h = \sum_{j=1}^{4} \Delta_j \phi_j$

AND TAKING $v: v = \phi_1^e ; \phi_2^e ; \phi_3^e$ and ϕ_4^e, we obtain the 4 equations that we need to obtain the four degree of freedom Δ_i.

$$\sum_{j=1}^{4} \underbrace{\left[\int_{x_e}^{x_{e+1}} \left(EI \frac{d^2\phi_i^e}{dx^2} \frac{d^2\phi_j^e}{dx^2} \right) dx \right]}_{K_{ij}^e} \Delta_j^e = \int_{x_e}^{x_{e+1}} \phi_i^e q \, dx + Q_i^e$$

$$K_{ij}^e \Delta_j^e = F_i^e + Q_i^e \quad \text{OR} \quad [K^e]\{\Delta^e\} = \{F^e\} + \{Q_i^e\}$$

where

$$K_{ij}^e = \int_{x_e}^{x_{e+1}} \left(EI \frac{d^2\phi_i^e}{dx^2} \frac{d^2\phi_j^e}{dx^2} \right) dx$$

$$F_i^e = \int_{x_e}^{x_{e+1}} \phi_i^e q \, dx + Q_i^e$$

IN MATRIX FORM

$$\begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e & K_{14}^e \\ K_{21}^e & K_{22}^e & K_{23}^e & K_{24}^e \\ K_{31}^e & K_{32}^e & K_{33}^e & K_{34}^e \\ K_{41}^e & K_{42}^e & K_{43}^e & K_{44}^e \end{bmatrix} \begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

DETERMINING THE INTEGRALS WE HAVE

$$[K^e] = \frac{2E_e I_e}{h_e^3} \begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix}$$

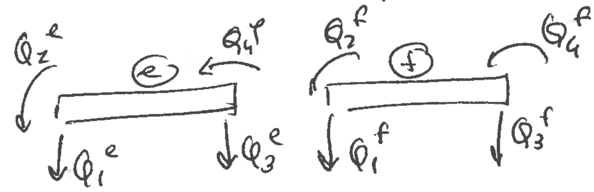
$$\{F^e\} = \frac{q_e h_e}{12} \begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} \rightarrow \text{for } q \text{ constant } \cancel{\text{uniform}} \text{ within an element.}$$

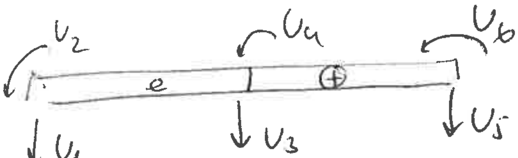
for any $q(x) \Rightarrow q_i^e = \int_{x_e}^{x_{e+1}} \phi_i^e q \, dx$.

CONECTIVITY OF THE ELEMENTS (ASSEMBLY)

$$Q_3^e + Q_1^f = F_0$$

$$Q_4^e + Q_2^f = M_0$$





$\Delta_1^e = U_1$

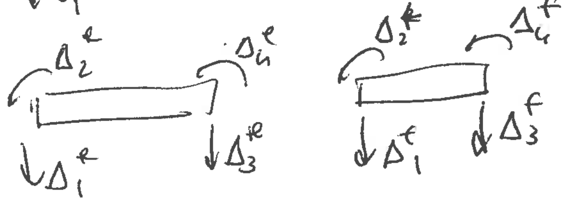
$\Delta_2^e = U_2$

$\Delta_3^e = \Delta_1^f = U_3$

$\Delta_4^e = \Delta_2^f = U_4$

$\Delta_3^2 = U_5$

$\Delta_4^2 = U_6$

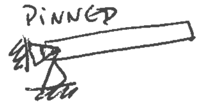
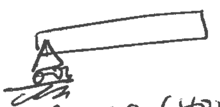
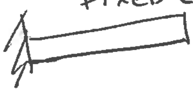


REMARK: TO ASSEMBLY THE ELEMENT WE CAN USE THE CONECTIVITY MATRIX:

ELEM. \ DF	Δ_1	Δ_2	Δ_3	Δ_4
1				
2				
⋮				

→ GLOBAL D.f.

BOUNDARY CONDITIONS: (see aceteto)

	DISPLACEMENT	FORCE
FREE 	NONE	ALL, AS SPECIFIED
PINNED 	$u=0$, $w=0$	MOMENT IS SPECIFIED
 ROLLER (VERTICAL)	$u=0$	TRANVERSE FORCE AND MOMENT SPEC.
 ROLLER (HORIZONTAL)	$w=0$	HORIZONTAL FORCE AND MOMENT
FIXED (OR CLAMPED) 	$u=0$, $w=0$, $dw/dx=0$	NONE

ELASTICALLY RESTRAINED

$$EI\,(d^2w/dx^2) + \mu\,\theta = M_0 \;;\; M_0 \text{ SPECIFIED}$$

$$EI\,(d^3w/dx^3) + Kw = Q_0 \;;\; Q_0 \text{ SPECIFIED.}$$

See page 246-247 to learn how to include the spring boundary condition

POST PROCESSING

THE SOLUTION IS GIVEN BY

$$w_h = \sum_{j=1}^{4} \Delta_j^e \phi_j^e$$

and the rotation by $-\dfrac{dw_h^e}{dx} = -\sum_{j=1}^{4} \Delta_j^e \dfrac{d\phi_j^e}{dx}$

the moment $-EI\dfrac{d^2w}{dx^2} = -EI \sum_{j=1}^{4} \Delta_j^e \dfrac{d^2\phi_j^e}{dx^2}$

and the stress ~~is~~ $\sigma_x = -\dfrac{M(x)\,z}{I} = -\cancel{Ez} \sum_{j=1}^{4} \Delta_j^e \dfrac{d^2\phi_j^e}{dx^2}$

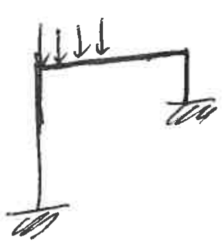
Remark: If EI constant the nodal solution is exact for any ~~all~~ type of distributed load $q(x)$.

The solution is exact if the load $q(x)$ is such, that the ~~exact~~ solution is cubic.

SOLVE PROBLEM.

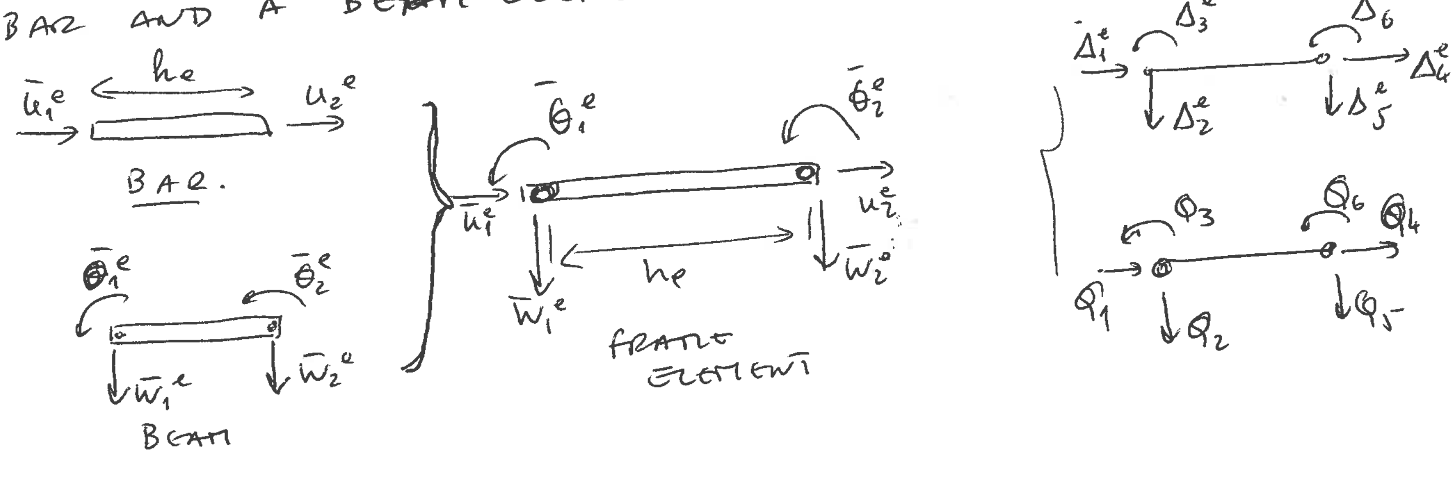
FRAMES

When structural members ~~are orie~~ that support bending are oriented in a plane the also have axial deformation. The structure, in this case, is called a FRAME structure.



TO ~~INCORPORATE~~ TAKE IN ACCOUNT BOTH THE BENDING AND AXIAL LOADING, WE CAN SUPERPOSE THESE TWO EFFECTS.

A FRAME ELEMENT IS OBTAINED BY THE SUPERPOSITION OF A BAR AND A BEAM ELEMENT.



THE STIFFNESS MATRIX IS A COMBINATION OF THE MATRIX FOR BAR AND BEAM ELEMENTS.

$$[\bar K]^e \{\Delta^e\} = \{\bar F\}^e \quad (6 \times 6$$

$$[\bar K]^e = \frac{2EI}{h^3}\begin{bmatrix} \mu & 0 & 0 & -\mu & 0 & 0 \\ 0 & 6 & -3h & 0 & -6 & -3h \\ 0 & -3h & 2h^2 & 0 & 3h & h^2 \\ -\mu & 0 & 0 & \mu & 0 & 0 \\ 0 & -6 & 3h & 0 & 6 & 3h \\ 0 & -3h & h^2 & 0 & 3h & 2h^2 \end{bmatrix} \quad \text{com} \quad \mu = \frac{Ah^2}{2I}$$

and the load vector for uniform $q(x)$ and $f(x)$

$$\{\bar F\}^e = \begin{Bmatrix} \frac{1}{2} fh \\ +\frac{1}{2} qh \\ -\frac{1}{12} qh^2 \\ \frac{1}{2} fh \\ \frac{1}{2} qh \\ \frac{1}{12} qh^2 \end{Bmatrix}$$

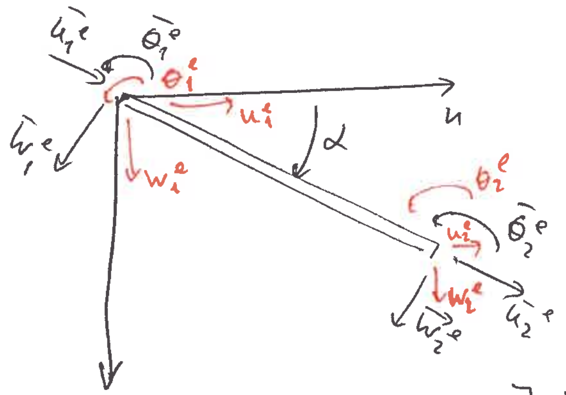
if q and f are not constant

thus

$$\bar f_i = \int_0^h f^e(\bar x)\, \underbrace{\psi_i^e(\bar x)}_{\to \text{lineare.}}\, d\bar x \qquad i = 1, 2$$

$$\bar q_i = \int_0^{h_e} q^e(\bar x)\, \phi_i^e(\bar x)\, d\bar x \qquad i = 1, 2, 3, 4.$$

For an oriented element:



THE TRANSFORMATION MATRIX IS:

$$\begin{Bmatrix} \bar x \\ \bar y \\ \bar z \end{Bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

and

$$\begin{Bmatrix} \bar u \\ \bar w \\ \bar \theta \end{Bmatrix}^e = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u \\ w \\ \theta \end{Bmatrix}^e$$

LOGO:

$$\begin{Bmatrix} \bar u_1 \\ \bar w_1 \\ \bar\theta_1 \\ \bar u_2 \\ \bar w_2 \\ \bar\theta_2 \end{Bmatrix} = \begin{bmatrix} \begin{matrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{matrix} \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

SIMILARLY TO THE BARS,

$$\{\bar{\Delta}^e\} = [T^e]\{\Delta^e\}$$

$$\{\bar{F}^e\} = [T^e]\{f\}^e$$

$$[\bar{K}]^e [T^e]^e \{\Delta\}^e = [T]^e \{f\}^e$$

$$[T]^T [\bar{K}]^e [T]^e \{\Delta\}^e = \{f\}^e \Rightarrow$$

$$\Rightarrow [K]^e \{\Delta\}^e = \{f\}^e$$

with, $[K]^e = [T]^T [\bar{K}]^e [T]^e$ and $\{f\}^e = [T]^T \{\bar{f}\}$

Thus

$$[K]^e = \frac{2EI}{h^3}\begin{bmatrix} \mu\cos^2\alpha + 6\sin^2\alpha & (\mu-6)\cos\alpha\sin\alpha & 3h\sin\alpha & -(\mu\cos^2\alpha + 6\sin^2\alpha) & -(\mu-6)\cos\alpha\sin\alpha & 3h\sin\alpha \\ & \mu\sin^2\alpha + 6\cos^2\alpha & -3h\cos\alpha & -(\mu-6)\sin\alpha\cos\alpha & -(\mu\sin^2\alpha + 6\cos^2\alpha) & -3h\cos\alpha \\ & & 2h^2 & -3h\sin\alpha & 3h\cos\alpha & h^2 \\ & & & \mu\cos^2\alpha + 6\sin^2\alpha & (\mu-6)\cos\alpha\sin\alpha & -3h\sin\alpha \\ & & & & \mu\sin^2\alpha + 6\cos^2\alpha & 3h\cos\alpha \\ & & & & & 2h^2 \end{bmatrix}$$

and $\{F\}^e = \begin{Bmatrix} \bar{F}_1\cos\alpha - \bar{F}_2\sin\alpha \\ \bar{F}_1\sin\alpha + \bar{F}_2\cos\alpha \\ \bar{F}_3 \\ \bar{F}_4\cos\alpha - \bar{F}_5\sin\alpha \\ \bar{F}_4\sin\alpha + \bar{F}_5\cos\alpha \\ \bar{F}_6 \end{Bmatrix}$

THE ASSEMBLY AND THE SOLUTION FOR THE SYSTEM OF EQUATION IS OBTAIN AS FOR THE PREVIOUS CASES.

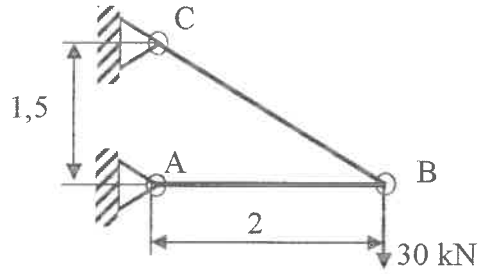
(PROBLEM)

Resolvere Problemen

Problema 10

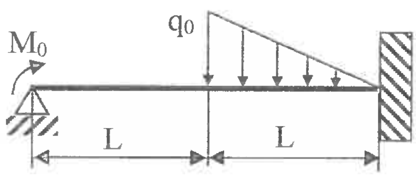
Para a estrutura representada na figura determine, utilizando o Método dos Elementos Finitos,

a) O deslocamento no ponto B.
b) As reacções nos apoios.



Problema 11

Considere a viga representada na figura para a qual $L=1$, $E=200\,GPa$ e $I=25\times10^{-5}m^4$, e onde $q_0=100\,Nm^{-1}$ e $M_0=100\,Nm$.

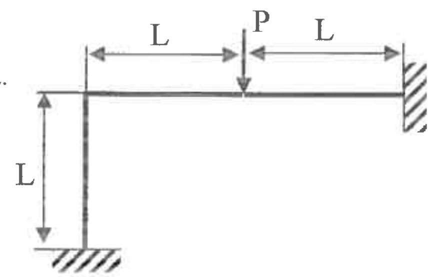


a) Utilizando dois elementos de viga de Euler-Bernouilli determine a matriz de rigidez global da estrutura.
b) Calcule o vector de forças global da estrutura.
c) Enuncie as condições de fronteira e calcule a solução do problema.
d) Calcule a reacção vertical no ponto A.

Problema 12

Para a estrutura representada na figura determine, utilizando o Método dos Elementos Finitos,

c) O deslocamento no ponto de aplicação da carga.
d) As reacções nos apoios.



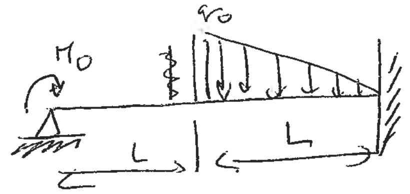
Considere a viga representada na figura para a qual $L=1$; $E=200$ GPa e $I=25\times10^{-5}\,m^4$ e onde $q_0=100\ Nm^{-1}$ e $M_0=100$ Nm.

a) Utilizando dois elementos de viga Euler-Bernoulli determine a matriz de rigidez global da estrutura.

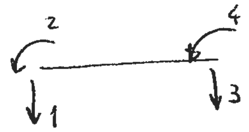
b) Calcule o vector de forças global da estrutura.

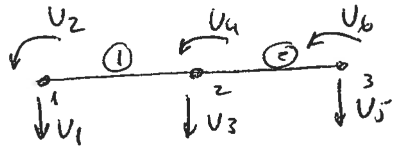
c) Imponha as condições de fronteira e calcule a solução do problema.

d) Calcule a reacção vertical no ponto A.



a)





Matriz de conectividades:

	1	2	3	4
1	1	2	3	4
2	3	4	5	6

$$K^e=\frac{2EI}{L^3}\begin{bmatrix}6 & -3h & -6 & -3h\\ -3h & 2h^2 & 3h & h^2\\ -6 & 3h & 6 & 3h\\ -3h & h^2 & 3h & 2h^2\end{bmatrix}$$

$K_{11}^G=K_{11}^{(1)}$ $\quad K_{12}^G=K_{12}^{(1)}$ $\quad K_{13}^G=K_{13}^{(1)}$ $\quad K_{14}^G=K_{14}^{(1)}$ $\quad K_{15}^G=0$ $\quad K_{16}^G=0$

$K_{22}^G=K_{22}^{(1)}$ $\quad K_{23}^G=K_{23}^{(1)}$ $\quad K_{24}^{(G)}=K_{24}^{(1)}$ $\quad K_{25}^G=0$ $\quad K_{26}^G=0$

$K_{33}^G=K_{33}^{(1)}+K_{11}^{(2)}$ $\quad K_{34}^G=K_{34}^{(1)}+K_{12}^{(2)}$ $\quad K_{35}^G=K_{13}^{(2)}$ $\quad K_{36}^G=K_{14}^{(2)}$

$K_{44}^G=K_{44}^{(1)}+K_{22}^{(2)}$ $\quad K_{45}^G=K_{23}^{(2)}$ $\quad K_{46}^G=K_{24}^{(2)}$

$K_{55}^G=K_{33}^{(2)}$ $\quad K_{56}^G=K_{34}^{(2)}$ $\quad K_{66}^G=K_{44}^{(2)}$

$$K^G=\frac{2EI}{1}\begin{bmatrix}6 & -3 & -6 & -3 & 0 & 0\\ -3 & 2 & 3 & 1 & 0 & 0\\ -6 & 3 & 12 & 0 & -6 & -3\\ -3 & 1 & 0 & 4 & 3 & 1\\ 0 & 0 & -6 & 3 & 6 & 3\\ 0 & 0 & -3 & 1 & 3 & 2\end{bmatrix}$$

(com $L=1$)

b) O vector de forças do elemento ① é nulo (antes de considerar o momento aplicado)

O vector de forças do elemento ② é calculado da seguinte forma:

$$F_i^e=\int_0^L \phi_i^e\, f(x)\,dx \qquad \text{com} \quad f(x)=+\frac{q_0}{L}(L-x)$$

e ϕ_i^e são as funções de forma:

$$\phi_1^e=1-3\left(\frac{x}{L}\right)^2+2\left(\frac{x}{L}\right)^3$$

$$\phi_2^e=-x\left(1-\frac{x}{L}\right)^2$$

$$\phi_3^e=3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3$$

$$\phi_4^e=-x\left[\left(\frac{x}{L}\right)^2-\frac{x}{L}\right]$$

$F_1 = \int_0^1 (1 - 3x^2 + 2x^3) * q_0(1-x)\,dx = \frac{7}{20} q_0$

$F_2 = \int_0^1 -x(1-x)^2 * (+q_0(1-x))\,dx = \frac{-q_0}{20}$

$F_3 = \int_0^1 (3x^2 - 2x^3)(q_0(1-x))\,dx = \frac{3q_0}{20}$

$F_4 = \int_0^1 -x(x^2-x)(q_0(1-x))\,dx = \frac{q_0}{30}$

$$f^{(2)} = \begin{Bmatrix} \frac{7}{20}q_0 \\ -\frac{q_0}{20} \\ \frac{3q_0}{20} \\ \frac{q_0}{30} \end{Bmatrix}$$

$$f^{GLOBAL} = \underbrace{\begin{Bmatrix} 0 \\ 0 \\ +\frac{7}{20}q_0 \\ -\frac{q_0}{20} \\ +\frac{3q_0}{20} \\ +\frac{q_0}{30} \end{Bmatrix}}_{DIST.} + \underbrace{\begin{Bmatrix} 0 \\ -M_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}}_{CONC.} + \underbrace{\begin{Bmatrix} R_A \\ 0 \\ 0 \\ 0 \\ R_B \\ M_B \end{Bmatrix}}_{REAC.}$$

$\int_0^1 -x(1+x^2-2x) * (q_0 - q_0 x)$

$\int_0^1 (-x + x^3 + 2x^2)(q_0 - q_0 x)$

$-q_0 x - q_0 x^3 + 2q_0 x^2 + q_0 x^2 + q_0 x^4 - 2q_0 x^3$

$(-q_0 x + 3q_0 x^2 - 3q_0 x^3 + q_0 x^4)$

$\left(-q_0 \frac{x^2}{2} + 3q_0\frac{x^3}{3} - 3q_0\frac{x^4}{4} + q_0\frac{x^5}{5}\right)$

$-\frac{1}{2} + 1 - \frac{3}{4} + \frac{1}{5}$

$-\frac{10}{20} + \frac{20}{20} - \frac{15}{20} + \frac{4}{20}$

c) Cond. fronteira

nó 1 $\quad w = 0 \quad (x=0) \Rightarrow U_1 = 0$

nó 3 $\quad w = 0 \quad (x = 2L) \Rightarrow U_5 = 0$

$\theta = 0 \quad U_6 = 0$

O sistema fica

$$2EI \begin{bmatrix} 6 & -3 & -6 & 3 & 0 & 0 \\ -3 & 2 & 3 & 1 & 0 & 0 \\ -6 & 3 & 12 & 0 & -6 & 3 \\ -3 & 1 & 0 & 4 & 3 & 1 \\ 0 & 0 & -6 & 3 & 6 & 3 \\ 0 & 0 & -3 & 1 & 3 & 2 \end{bmatrix} \begin{Bmatrix} U_1 = 0 \\ U_2 \\ U_3 \\ U_4 \\ U_5 = 0 \\ U_6 = 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -100 \\ +35 \\ -5 \\ +15 \\ +10/3 \end{Bmatrix} + \begin{Bmatrix} R_A \\ \\ \\ \\ R_B \\ M_B \end{Bmatrix}$$

$$2EI \begin{bmatrix} 2 & 3 & 1 \\ 3 & 12 & 0 \\ 1 & 0 & 4 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} -100 \\ +35 \\ -5 \end{Bmatrix} \Rightarrow \begin{array}{l} U_2 = -0.1075 \times 10^{-5} \\ U_3 = +0.298 \times 10^{-6} \\ U_4 = 0.256 \times 10^{-6} \end{array}$$

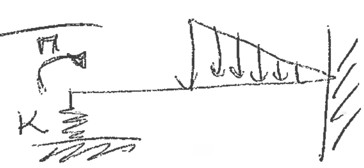
d) De equação 1

$2EI(6 \times 0 - 3 \times (-0.1075 \times 10^{-5}) - 6 \times 0.298 \times 10^{-6} + 3 \times 0.256 \times 10^{-6} + 0 + \ldots$

$= R_A$

$R_A = +66.90\ N$

T.P.C. Resolver com



(VER PÁG 247)

Matriz de rigidez para o elemento Viga-Barra (Viga Euler-Bernoulli).
Esta matriz substitui a matriz 5.4.10a (pág. 278) da 3ª edição do livro.

$$K^e = \frac{2EI}{h^3}\left[\begin{array}{ccc:ccc} \mu c^2+6s^2 & (\mu-6)cs & 3hs & -(\mu c^2+6s^2) & -(\mu-6)cs & 3hs \\ & \mu s^2+6c^2 & -3hc & -(\mu-6)cs & -(\mu s^2+6c^2) & -3hc \\ & & 2h^2 & -3hs & 3hc & h^2 \\ \hdashline & & & \mu c^2+6s^2 & (\mu-6)cs & -3hs \\ & & & & \mu s^2+6c^2 & 3hc \\ & & & & & 2h^2 \end{array}\right]$$

$$\mu = \frac{Ah^2}{2I}, \quad c = \cos(\alpha), \quad s = \sin(\alpha)$$