Bone Tissue Mechanics

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PART 4

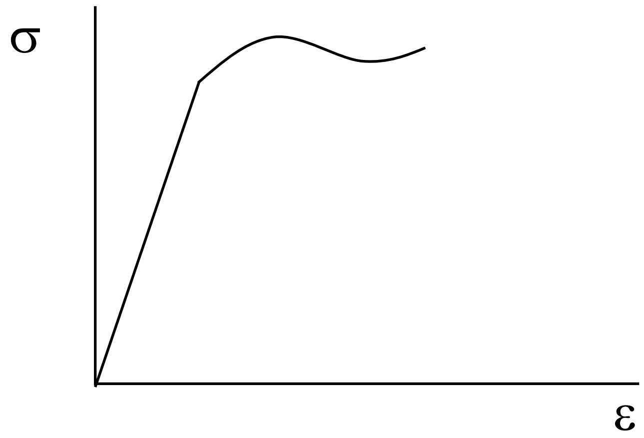
Generalized Hooke's Law

For a uniaxial state of stress the relationship between stress and strain (constitutive law) is given by:

$$\sigma = E\, \varepsilon$$

Similarly for shear we have:

$$\tau = G\, \gamma$$



Generalized Hooke's Law

For a generalized state of stress and/or non isotropic materials the Hooke's law is given by:

$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$ $\quad$ i,j,k,l=1,2,3 (3D) - sum convenction

C_{ijkl} = tensor of the elastic properties (fourth-rank tensor)

The tensor of the elastic properties obey to the coordinate transformation law:

$$C'_{ijkl} = R_{im}\, R_{jn}\, R_{kp}\, R_{lq}\, C_{mnpq}$$

where ***R*** is the transformation of coordinates matrix

The number of components C_{ijkl} is ($3\times 3\times 3\times 3 =$) 81,

but by symmetry, $C_{ijkl} = C_{jikl}$, $C_{ijkl} = C_{ijlk}$, $C_{ijkl} = C_{klij}$,

There are only 21 independent coefficients (anisotropic material)

For example ($\sigma_{ij}=C_{ijkl}\,\varepsilon_{kl}$)

$$\sigma_{11}=C_{1111}\varepsilon_{11}+C_{1112}\varepsilon_{12}+C_{1113}\varepsilon_{13}+$$
$$+C_{1121}\varepsilon_{21}+C_{1122}\varepsilon_{22}+C_{1123}\varepsilon_{23}+$$
$$+C_{1131}\varepsilon_{31}+C_{1132}\varepsilon_{32}+C_{1133}\varepsilon_{33}$$

$$\sigma_{11}= C_{1111}\varepsilon_{11}+C_{1122}\varepsilon_{22}+C_{1133}\varepsilon_{33}+2\times C_{1123}\varepsilon_{23}+2\times C_{1113}\varepsilon_{13}+2\times C_{1112}\varepsilon_{12}$$

The constitutive law for elastic materials can be written in the matrix form as:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ & & & C_{2323} & C_{2313} & C_{2312} \\ & & & & C_{1313} & C_{1312} \\ \text{sim.} & & & & & C_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}$$

Remark:
- the order of the terms depends on the author
- $2\varepsilon_{23}=\gamma_{23}$, $2\varepsilon_{13}=\gamma_{13}$, $2\varepsilon_{12}=\gamma_{12}$

A simplified way to write the previous relation is:

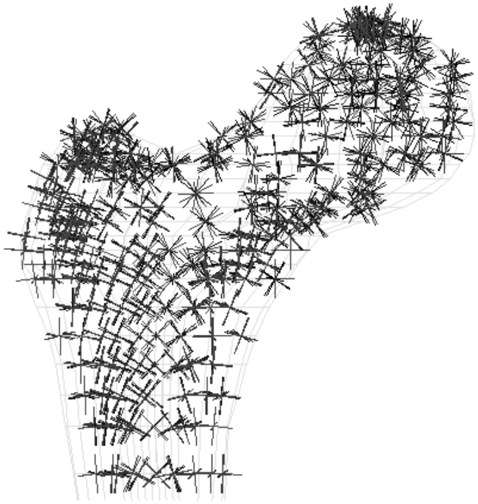
$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ \text{sim.} & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

where:
11→1 , 22→2 , 33→3
23→4 , 13→5 , 12→6
and,
$2\varepsilon_{23}=\varepsilon_4$, $2\varepsilon_{13}=\varepsilon_5$, $2\varepsilon_{12}=\varepsilon_6$

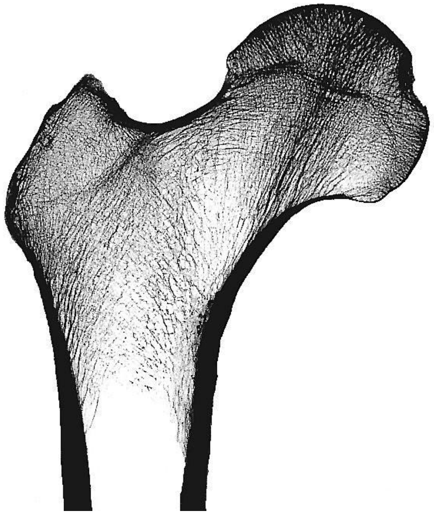
- The 21 independent coefficients are for anisotropic material

- By <u>material symmetry</u> this number is reduced. Some of them as zero and some relations can be derived for others.

- For na isotropic material we have only 2 independent constants: E and G, with $E=2G(1+\nu)$



Fernandes, Rodrigues and Jacobs (1999)

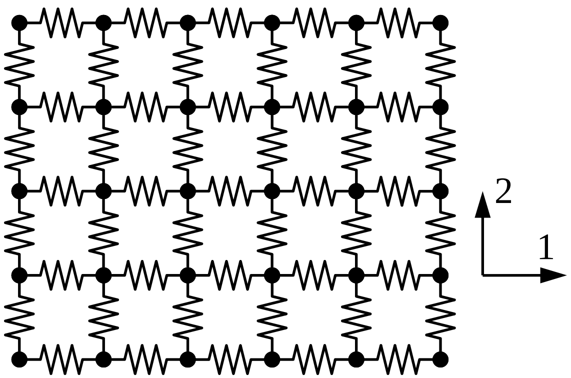


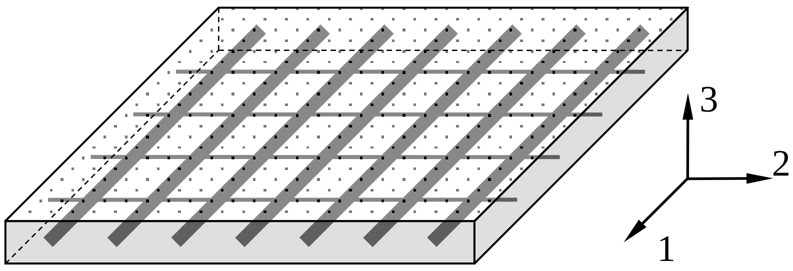
• Bone, due to is nature and constitution (it can be considered a composite material) is in general non isotropic. There are region where it behaves as anisotropic, regions where it is orthotropic and even region where the behavior approaches the isotropy.

Material symmetry

- in the case of material symmetry the tensor of elastic coefficients simplifies.
- if the material has 3 planes of symmetry, the material is **orthotropic** (9 coeficients).

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & C_{22} & C_{23} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & C_{33} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & & C_{44} & \mathbf{0} & \mathbf{0} \\ & & & & C_{55} & \mathbf{0} \\ \text{sim.} & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$



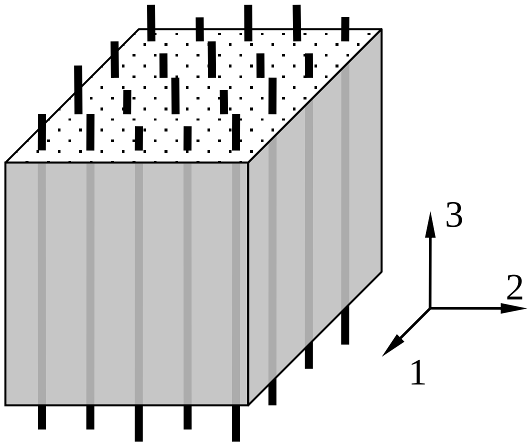


geometric symmetry ⇒ material symmetry

- At a macroscale bone is often modeled as na orthotropic material

• when we have a value for the stiffness in one direction and another for all transverse directions the material is **transversely isotropic** (5 coefficients)

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ \text{sim.} & & & & & \frac{1}{2}\left(C_{11} - C_{12}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$



• compact bone is usually modeled as a transversly isotropic material.

• when there are infinite plane of symmetry, the material is isotropic (2 coeficiente $\rightarrow C_{11}$ e C_{12} or E e ν or E e G)

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \frac{1}{2}\left(C_{11}-C_{12}\right) & 0 & 0 \\ & & & & \frac{1}{2}\left(C_{11}-C_{12}\right) & 0 \\ \text{sim.} & & & & & \frac{1}{2}\left(C_{11}-C_{12}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

nota: $C_{11} = \dfrac{E\left(1-\nu\right)}{\left(1+\nu\right)\left(1-2\nu\right)}$, $C_{12} = \dfrac{E\nu}{\left(1+\nu\right)\left(1-2\nu\right)}$, $\dfrac{1}{2}\left(C_{11}-C_{12}\right) = \dfrac{E}{2\left(1+\nu\right)} = G$

• the macroscale behavior of bone is not isotropic (even if it can has isotropic characteristics at some points)

• however, in some mathematical and computational models bone is considered isotropic (to avoid more complex analysis or due to lack of data)

Compliance matrix

$$\{\sigma\} = [C]\{\varepsilon\} \Rightarrow \{\varepsilon\} = [C]^{-1}\{\sigma\} \Rightarrow \{\varepsilon\} = [S]\{\sigma\} \quad \rightarrow \quad [S] = [C]^{-1}$$

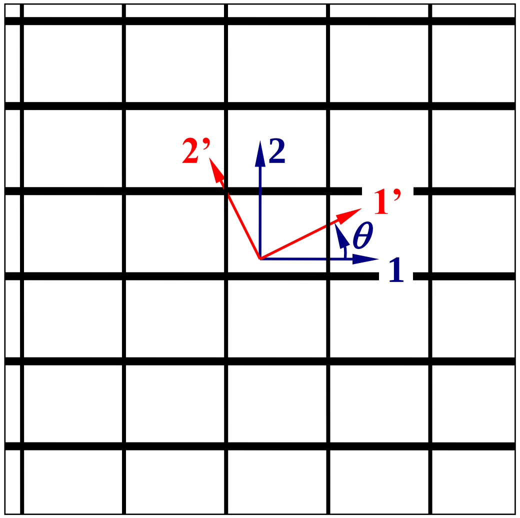
- [S] is the compliance (inverse of stiffness),
- [S] is analog to [C] in terms of structures and null coefficients

example: isotropic material

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ & 1/E & -\nu/E & 0 & 0 & 0 \\ & & 1/E & 0 & 0 & 0 \\ & & & 1/G & 0 & 0 \\ & & & & 1/G & 0 \\ \text{sim.} & & & & & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

Orthotropic material (2D) – plane stress

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ & S_{22} & 0 \\ \text{sim.} & & S_{66} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}$$



$$\boldsymbol{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ & C_{22} & 0 \\ \text{sim.} & & C_{66} \end{bmatrix}$$

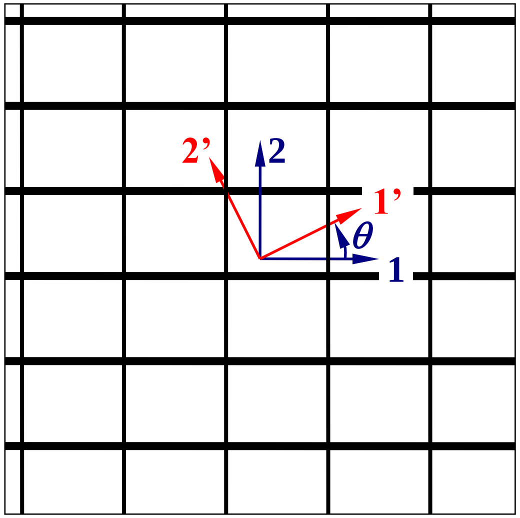
- Usually the data are in terms of Young Modulus and Poisson ratio

Experimental determination of the elastic coefficients

- orthotropic material: 9 independents coefficients
- assuming 2D: 4 independents coefficients → $E_1, E_2, \nu_{12}, G_{12}$

note: by tensor symmetry, $\nu_{12} / E_1 = \nu_{21} / E_2$)

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ & S_{22} & 0 \\ \text{sim.} & & S_{66} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}$$



Experimental determination of the elastic coefficients

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ & S_{22} & 0 \\ \text{sim.} & & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

- Mechanical tests controling the applied stress and measuring strains.

Test 1: Uniaxial stress test, apply σ_1, read ε_1 and ε_2

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} \frac{\sigma_1}{E_1} \\ -\frac{\nu_{12}\sigma_1}{E_1} \\ 0 \end{Bmatrix} \Rightarrow \begin{cases} E_1 = \frac{\sigma_1}{\varepsilon_1} \\ \nu_{12} = -\frac{E_1 \varepsilon_2}{\sigma_1} \end{cases}$$

Experimental determination of the elastic coefficients

Test 2: Uniaxial stress test, apply σ_2, read ε_2 and ε_1)

$$\ldots \Rightarrow \begin{cases} E_2 = \dfrac{\sigma_2}{\varepsilon_2} \\ \nu_{21} = -\dfrac{E_2 \varepsilon_1}{\sigma_2} \quad \leftarrow \text{verificação} \end{cases}$$

Test 3: Shear test (torsion), apply σ_6 (= σ_{12}), read ε_6 (= 2 ε_{12} = γ_{12})

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{E_1} & -\dfrac{\nu_{21}}{E_2} & 0 \\ -\dfrac{\nu_{12}}{E_1} & \dfrac{1}{E_2} & 0 \\ 0 & 0 & \dfrac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \sigma_6 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{\sigma_6}{G_{12}} \end{Bmatrix} \Rightarrow G_{12} = \frac{\sigma_6}{\varepsilon_6}$$

Transformation of coordinates

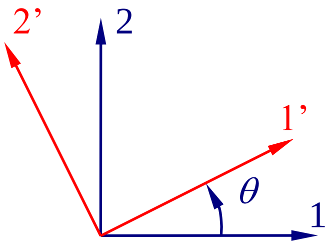
- for simplicity, let us assume a 2D case

Going back to the tensor form of ***C*** (or of ***S***)

$$C'_{ijkl} = R_{im} R_{jn} R_{kp} R_{lq} C_{mnpq} \qquad i,j,k,l,m,n,p,q = 1,2 \text{ (2D)}$$

where ***R*** is the rotation matrix that tranform the reference system 1,2 to 1’,2’

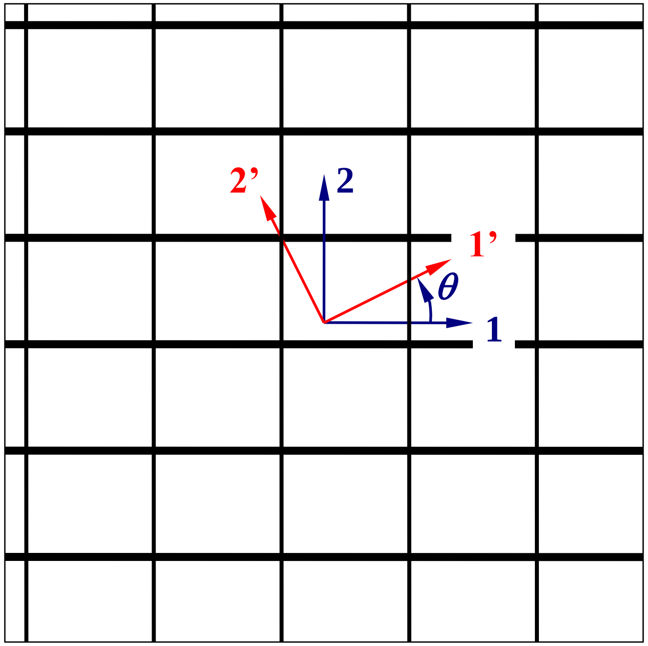
$$[R] = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$



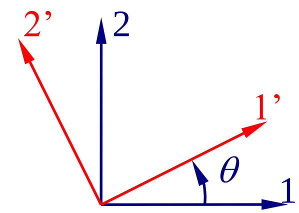
Orthotropic material (2D)

$$\boldsymbol{S} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ & S_{22} & 0 \\ \text{sim.} & & S_{66} \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ & C_{22} & 0 \\ \text{sim.} & & C_{66} \end{bmatrix}$$



Orthotropic material (2D) – transformation of coordinates



$$C'_{ijkl} = R_{im} R_{jn} R_{kp} R_{lq} C_{mnpq} \qquad i,j,k,l,m,n,p,q=1,2$$

$$\boldsymbol{C'} = \begin{bmatrix} C'_{11} & C'_{12} & C'_{16} \\ & C'_{22} & C'_{26} \\ \text{sim.} & & C'_{66} \end{bmatrix} \qquad [R] = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ & C_{22} & 0 \\ \text{sim.} & & C_{66} \end{bmatrix}$$

$$C'_{11}=\cos^4(\theta).C_{11}+2.\sin^2(\theta).\cos^2(\theta).C_{12}+\sin^4(\theta).C_{22}+4.\sin^2(\theta).\cos^2(\theta).C_{66}$$

$$C'_{12}= \ldots$$

$$C'_{16}=-\sin(\theta).\cos(\theta).[\cos^2(\theta).C_{11}-\cos(2\theta).C_{12}-\sin^2(\theta).C_{22}-2.\cos(2\theta).C_{66}]$$

$$C'_{22}= \ldots$$

$$C'_{26}=\sin(\theta).\cos(\theta).[-\sin^2(\theta).C_{11}-\cos(2\theta).C_{12}+\cos^2(\theta).C_{22}-2.\cos(2\theta).C_{66}]$$

$$C'_{66}=\sin^2(\theta).\cos^2(\theta).(C_{11}-2.C_{12}+C_{22})+\cos^2(2\theta)\cdot C_{66}$$

- for a given reference system (not the principal material axes), the matrix for the elastic coefficients of na orthotropic material is full (without zeros)

Principal directions for stress and strain

• for a principal referencial (the axes are the directions of orthotropy (2D)

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ & S_{22} & 0 \\ \text{sim.} & & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

• if we assume a principal state of stress $\{\sigma\}=\{a,b,0\}$. The strain field is,

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} a \\ b \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} S_{11}.a + S_{12}.b \\ S_{12}.a + S_{22}.b \\ 0 \end{Bmatrix} \quad \rightarrow \quad \varepsilon_6 = 0$$

• if the principal stress directions coincide with the principal axes of the material (principal material directions), thus the principal directions for stress are also coincident with the principal directions for strain.

Principal directions for stress and strain

• for a given reference system (not the principal axes , not the directions of orthotropy)

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S'_{11} & S'_{12} & S'_{16} \\ & S'_{22} & S'_{26} \\ \text{sim.} & & S'_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

• assuming a principal state of stress $\{\sigma\}=\{a,b,0\}$. The resulting strain field is:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S'_{11} & S'_{12} & S'_{16} \\ S'_{12} & S'_{22} & S'_{26} \\ S'_{16} & S'_{26} & S'_{66} \end{bmatrix} \begin{Bmatrix} a \\ b \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} S'_{11}.a + S'_{12}.b \\ S'_{12}.a + S'_{22}.b \\ S'_{16}.a + S'_{26}.b \end{Bmatrix} \quad \rightarrow \quad \varepsilon_6 \neq 0$$

• if the principal stress directions **do not** coincide with the principal material directions (directions of orthotropy), thus the principal stress directions **do not** coincide with the principal strain directions.

Problem – strain energy density

Determine the strain energy density, $u = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$, for an orthotropic material when the direction of orthotropy 1 make an angle of 30° with the horizontal, and it is subjected to a uniaxial tensile stress σ_x=10 MPa

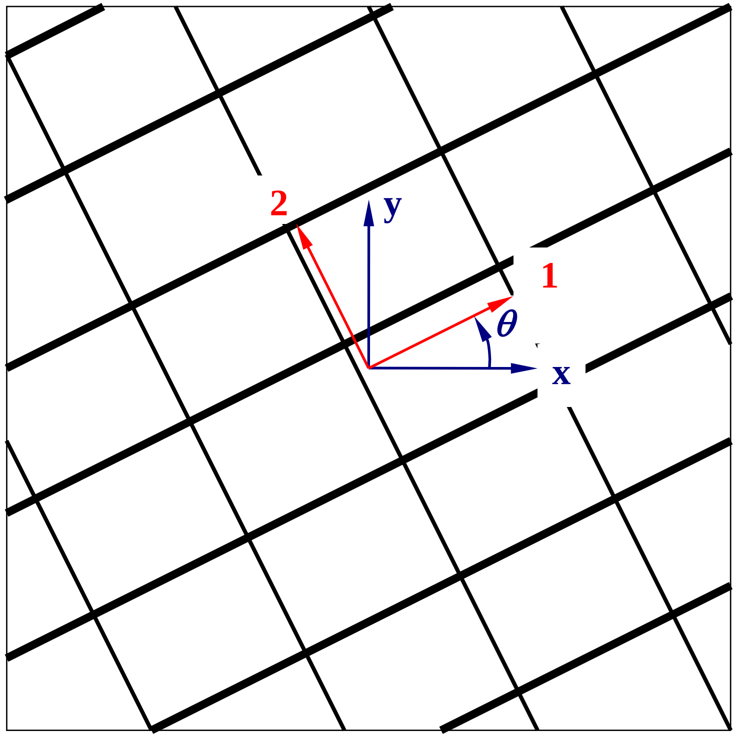
Material data:

E_1=20GPa

E_2=10GPa

ν_{12}=0.4

G_{12}=3.5GPa



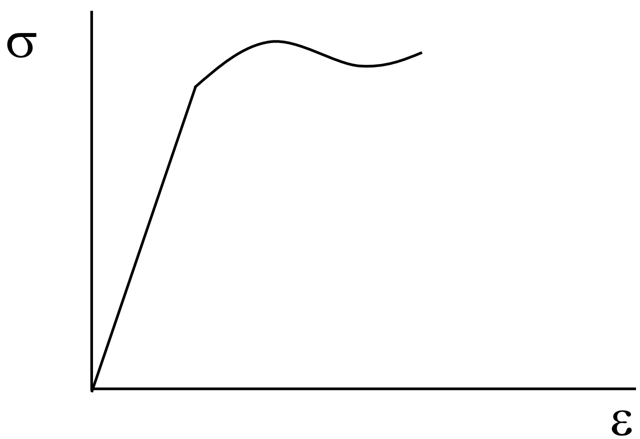
Elastic strain energy

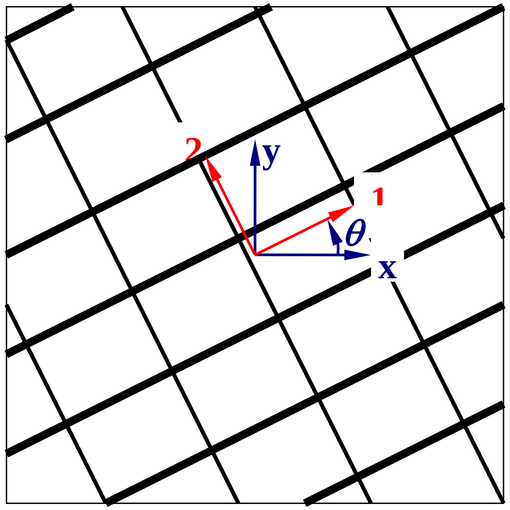
$$u = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left(\sigma_{11}\varepsilon_{11} + \sigma_{12}\varepsilon_{12} + \sigma_{21}\varepsilon_{21} + \sigma_{22}\varepsilon_{22}\right)$$

$$= \frac{1}{2}\left(\sigma_{xx}\varepsilon_{xx} + \sigma_{xy}\varepsilon_{xy} + \sigma_{yx}\varepsilon_{yx} + \sigma_{yy}\varepsilon_{yy}\right)$$

• To compute the strain energy, we need to know the stress and strain field in a given reference system.

→ Let's do the calculus in the material referential (referential 12)





Calculus of stress in the material referential

$$\sigma_{ij} = R_{ik} R_{jm} \sigma_{km} \rightarrow [\sigma]_{12} = [R][\sigma]_{xy} [R]^T$$

$$[\sigma]_{xy} = \begin{bmatrix} \sigma_{xx} & 0 \\ 0 & 0 \end{bmatrix}$$

$$[R] = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(30) & \sin(30) \\ -\sin(30) & \cos(30) \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

Calculus of stress in the material referential

$$[\sigma]_{12} = [R][\sigma]_{xy}[R]^T = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \sqrt{3}\sigma_{xx}/2 & -\sigma_{xx}/2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3\sigma_{xx}/4 & -\sqrt{3}\sigma_{xx}/4 \\ -\sqrt{3}\sigma_{xx}/4 & \sigma_{xx}/4 \end{bmatrix}$$

$$[\sigma]_{12} = \begin{bmatrix} 3\sigma_{xx}/4 & -\sqrt{3}\sigma_{xx}/4 \\ -\sqrt{3}\sigma_{xx}/4 & \sigma_{xx}/4 \end{bmatrix} \rightarrow \{\sigma\}_{12} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{Bmatrix} 3\sigma_{xx}/4 \\ \sigma_{xx}/4 \\ -\sqrt{3}\sigma_{xx}/4 \end{Bmatrix} = \begin{Bmatrix} 7.5 \\ 2.5 \\ -4.3 \end{Bmatrix} \text{ MPa}$$

Calculus of strain in the material referential

$$\{\varepsilon\}_{12} = [S]_{12}\{\sigma\}_{12}$$

$$[S]_{12} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} = \begin{bmatrix} \frac{1}{20} & -\frac{0.4}{20} & 0 \\ -\frac{0.4}{20} & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{3.5} \end{bmatrix} \times 10^{-3} = \begin{bmatrix} 0.05 & -0.02 & 0 \\ -0.02 & 0.1 & 0 \\ & & 0.286 \end{bmatrix} \times 10^{-3}$$

$$\{\varepsilon\}_{12} = [S]_{12}\{\sigma\}_{12} = \begin{bmatrix} 0.05 & -0.02 & 0 \\ -0.02 & 0.1 & 0 \\ & & 0.286 \end{bmatrix} \begin{Bmatrix} 7.5 \\ 2.5 \\ -4.3 \end{Bmatrix} \times 10^{-3} = \begin{Bmatrix} 0.325 \\ 0.1 \\ -1.23 \end{Bmatrix} \times 10^{-3}$$

Calculus of strain energy

$$u = \frac{1}{2}\left(\sigma_{11}\varepsilon_{11} + \sigma_{12}\varepsilon_{12} + \sigma_{21}\varepsilon_{21} + \sigma_{22}\varepsilon_{22}\right)$$

$$u = \frac{1}{2}\left(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + 2\sigma_{12}\varepsilon_{12}\right)$$

$$u = \frac{1}{2}\left(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{12}\gamma_{12}\right)$$

$$u = \frac{1}{2}\left(\sigma_{1}\varepsilon_{1} + \sigma_{2}\varepsilon_{2} + \sigma_{6}\gamma_{6}\right)$$

$$u = \frac{1}{2}\left(7.5\times 0.325 + 2.5\times 0.1 + (-4.3)\times(-1.23)\right)\times 10^{-3}$$

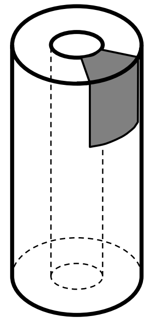
$$u = 3.99\times 10^{-3}\,\mathrm{N.mm}^{-2}$$

Mechanical properties for compact bone

TABLE 4.1. Typical mechanical properties for cortical bone

Property	Human	Bovine
Elastic modulus, GPa		
Longitudinal	17.4	20.4
Transverse	9.6	11.7
Bending	14.8[a]	19.9[b]
Shear modulus, GPa	3.51	4.14
Poisson's ratio	0.39	0.36
Tensile yield stress, MPa		
Longitudinal	115	141
Transverse	—	—
Compressive yield stress, MPa		
Longitudinal	182	196
Transverse	121	150
Shear yield stress, MPa	54	57
Tensile ultimate stress, MPa		
Longitudinal	133	156
Transverse	51	50
Compressive ultimate stress, MPa		
Longitudinal	195	237
Transverse	133	178
Shear ultimate stress, MPa	69	73
Bending ultimate stress, MPa	208.6[a]	223.8[b]
Tensile ultimate strain		
Longitudinal	0.0293	0.0072
Transverse	0.0324	0.0067
Compressive ultimate strain		
Longitudinal	0.0220	0.0253
Transverse	0.0462	0.0517
Shear ultimate strain	0.33	0.39
Bending ultimate strain	—	0.0178[b]

Human data are for adult femur and tibia; bovine data are for femur.
Data compiled from Cowin (1989), pp. 102, 103, 111–113, except as indicated.
[a]From Currey and Butler (1975); adult femur, three-point bending.
[b]From Martin and Boardman (1993); tibia, three-point bending.



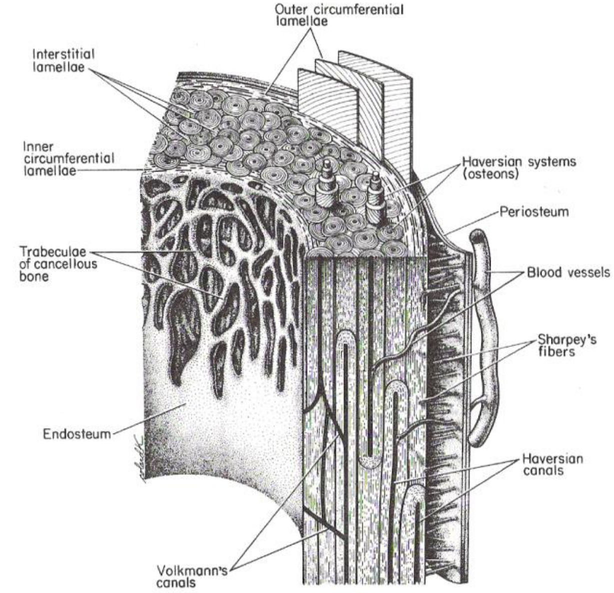
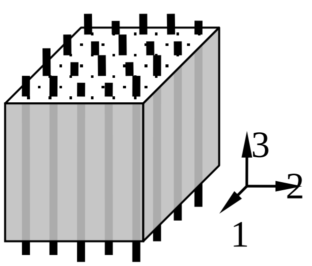


FIGURE 2.1. Sketch of some important features of a typical long bone. (After Benninghoff, 1949.)

?

transversely isotropic

$$\boldsymbol{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ \text{sim.} & & & & & \frac{1}{2}(C_{11}-C_{12}) \end{bmatrix}$$



example – bone with implant

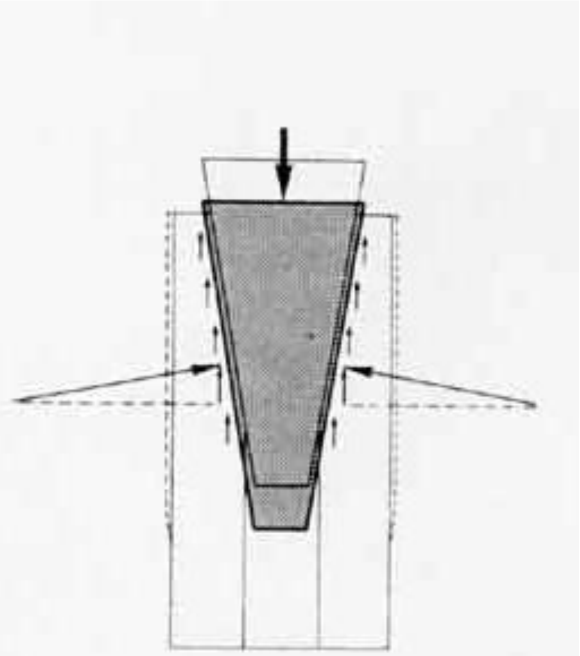


TABLE 4.1. Typical mechanical properties for cortical bone

Property	Human	Bovine
Tensile ultimate stress, MPa		
Longitudinal	133	156
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Longitudinal	195	237
Transverse	133	178
Shear ultimate stress, MPa	69	73
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- usual loads do not lead to high transverse normal stresses
- however, there are situations where the transverse normal stress can be high, for instance the introduction of an implant can induce this kind of stress and lead to bone failure.

Failure criteria for orthotropic materials

• for orthotropic materials (or transversely isotropic), failure criteria such as the Von Mises criterion are not adequate.

• a failure criterion often used for this type of materials is the Tsai-Wu criterion. (it is applied for composite materials – laminates reinforced with fibers – it is adequate for cortical bone)

• theTsai-Wu criterion says the the failure occurs when the stress level, described by the following expression, is greater or equal to 1

$$F_1\sigma_1 + F_2\sigma_2 + F_3\sigma_3 + F_{11}(\sigma_1)^2 + F_{22}(\sigma_2)^2 + F_{33}(\sigma_3)^2 +$$
$$+2F_{12}\sigma_1\sigma_2 + 2F_{13}\sigma_1\sigma_3 + 2F_{23}\sigma_2\sigma_3 + F_{44}(\sigma_4)^2 + F_{55}(\sigma_5)^2 + F_{66}(\sigma_6)^2 = 1$$

where $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6$ are the stresses $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{13}$ **in the material reference system**, and the 12 coefficients F_i, F_{ij} are determined experimentally

Failure criteria for orthotropic materials

How to obtain the coefficients for the Tsai-Wu criterion?

Example

• for instance, doing uniaxial tension and compression tests in direction 1 (direction of orthotropy) we have:

tension: $F_1\sigma_1^{tf} + F_{11}(\sigma_1^{tf})^2=1$

compression: $F_1\sigma_1^{cf} + F_{11}(\sigma_1^{cf})^2=1$

where σ_1^{tf} e σ_1^{cf} are the ultimate values for tension and compression, respectively

Solving the system for the two unknowns:

$F_1=1/\sigma_1^{tf}+1/\sigma_1^{cf}$ $\qquad$ $F_{11}=-1/(\sigma_1^{tf}\,\sigma_1^{cf})$

• similar test can be done for the remaining directions as well as shear tests to obtain the coefficients F_i e F_{ii}

• the coefficients F_{ij}, $i\neq j$, must be obtained by biaxial tests.

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